cvefp.tex dias-stp-01-04

Covariant Quantum Green's Function for an Accelerated Particle

T. Garavaglia*

Institiúid Árd-léinn Bhaile Átha Cliath, Baile Átha Cliath 4, Éire†

Covariant relativistic quantum theory is used to study the covariant Green's function, which can be used to determine the proper time evolved wave functions that are solutions to the covariant Schrödinger type equation for a massive spin zero particle. The concept of covariant action is used to obtain the Green's function for an accelerated relativistic particle.

PACS: 04.60, 03.65, 03.65P, 03.30

---

* Email address: bronco@stp.dias.ie

† Also *Institiúid Teicneolaíochta Bhaile Átha Cliath.*

Covariant relativistic quantum theory has been described in [1], and the methods that are developed there are used to find the proper time covariant Green's function for a relativistic spin zero particle of mass m that undergoes uniform acceleration. Covariant relativistic quantum theory is a natural extension of the concepts of classical covariant relativistic particle dynamics [2]. Proper time covariant methods have been used previously in the context of relativistic wave equations [3], and quantum field theory [4]. The Lorentz invariant parameter that is relevant for dynamical development in classical covariant mechanics is proper distance s, proper time times the speed of light, and the invariant parameter that characterizes the rest energy of a particle is its mass. The evolution of quantum states with respect to the parameter s is found in terms of the covariant Green's function. This Green's function is found with the aid of the concept of covariant action. The resulting Green's function has a practical application in the study of the quantum state proper time evolution of, for example, an accelerated spin zero heavy ion.

The conventions used are those of [1], along with the units $\hbar = c = 1$. The space-time contravariant coordinate four-vector is defined as $q^\mu = (q^0, q^1, q^2, q^3) = (ct, x, y, z)$, and the dimensionless form is found by dividing the components with the fundamental units of length q_s. The Einstein summation convention is used throughout, and the covariant components are found from $q_\mu = g_{\mu\nu} q^\nu$, with the non-zero components of the metric tensor given by $(g_{00}, g_{11}, g_{22}, g_{33}) = (1, -1, -1, -1)$. The scalar product of two four-vectors is $q \cdot p = q_\mu p^\mu$, and the space-time measure is $ds = \sqrt{dq \cdot dq}$. The four-momentum for a free particle is $p^\mu = m u^\mu = (p^0, \vec{p}) = (\gamma m, \gamma \vec{\beta} m)$, with $\vec{\beta} = d\vec{q}/dq^0$, $\gamma = 1/\sqrt{1 - \vec{\beta} \cdot \vec{\beta}}$, $p \cdot p = m^2$, and $u^\mu = dq^\mu/ds$.

Covariant action is defined as

$$\mathcal{S} = \int \mathcal{L}\left(q, u\right) ds, \tag{1}$$

where $\mathcal{L}(q, u)$ is a Lorentz invariant Lagrangian. The requirement that the action be an extremum under a variation of the coordinates q^μ leads to the covariant Euler-Lagrange equation

$$\frac{d \frac{\partial \mathcal{L}}{\partial u^\mu}}{ds} - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0. \tag{2}$$

The covariant Lagrangian for a particle of mass m and charge e interacting with a four-vector field $A^\mu(q)$ is

$$\mathcal{L} = \frac{m}{2} u^2 - eA \cdot u, \tag{3}$$

and the generalized four-momentum is

$$p_\mu = \frac{\partial \mathcal{L}}{\partial u^\mu} = m u_\mu - e A_\mu. \tag{4}$$

The equation of motion is found from Eq. (2) to be

$$\frac{dp_\mu}{ds} + \partial_\mu e A \cdot u = 0, \tag{5}$$

where $\partial_\mu = \partial / \partial q^\mu$.

From the covariant Lagrangian Eq. (3), the covariant Hamiltonian is defined as

$$\mathcal{H} = p \cdot u - \mathcal{L} = \frac{\left(p + eA\right)^2}{2m}. \tag{6}$$

The properties of the covariant action can be used to determine the covariant quantum Green's function for an accelerated spin zero particle of unit mass. For the four-vector potential $eA^\mu = (-V(\vec{q}), 0, 0, 0)$, the differential equation for the covariant action is

$$\frac{1}{2}\left(\left(\frac{\partial S}{\partial q^0} - V\left(\vec{q}\right)\right)^2 - \vec{\nabla} S \cdot \vec{\nabla} S\right) + \frac{\partial S}{\partial s} = 0. \tag{7}$$

The four-momentum associated with the covariant action is

$$p_\mu = \frac{\partial S}{\partial q^\mu}. \tag{8}$$

For a free particle of unit mass the solution to Eq. (7) is

$$S = \frac{(q-q')\cdot(q-q')}{2s}. \tag{9}$$

For a time independent potential $V(z)$, p^0 is a constant, and the covariant action can be written as

$$S\left(q,s\right) = S_0\left(z,p^0,s\right) - \frac{y^2}{2s} - \frac{x^2}{2s} + p^0 t; \tag{10}$$

hence, Eq. (7) becomes

$$\frac{1}{2}\left(\left(p^0 - V\left(z\right)\right)^2 - \left(\frac{\partial S_0\left(z,p^0,s\right)}{\partial z}\right)^2\right) + \frac{\partial S_0\left(z,p^0,s\right)}{\partial s} = 0. \tag{11}$$

From Eq. (4) and Eq. (5), the equations of motion become

$$\begin{aligned} &\frac{dt}{ds} = p^0 - V\left(z\right), \quad \frac{dp^0}{ds} = 0, \\ &\frac{dz}{ds} = p^3, \quad \frac{dp^3}{ds} = -\left(p^0 - V\left(z\right)\right)\frac{\partial V\left(z\right)}{\partial z}. \end{aligned} \tag{12}$$

A particle with constant acceleration results from the potential $V(z) = -gz$, where g is a constant. With the replacements $q^\mu \to q^\mu/\sqrt{g}$, $p^\mu \to p^\mu\sqrt{g}$, and $s \to s/g$, the equations of motion become

$$\frac{dt}{ds} = p^0 + z = \xi, \quad \frac{d^2\xi}{ds^2} = \xi. \tag{13}$$

The solutions to these equations are

$$\begin{aligned} \xi\left(s\right) &= \xi\left(0\right)\cosh\left(s\right) + b\sinh\left(s\right) \\ t\left(s\right) &= \xi\left(0\right)\sinh\left(s\right) + b\left(\cosh\left(s\right) - 1\right) + t\left(0\right). \end{aligned} \tag{14}$$

The action function $S_0(\xi, s)$ is now found from

$$S_0\left(\xi,s\right) = \int_0^s \mathcal{L}_0 ds, = -\int_0^s \frac{1}{2}\left(\left(\frac{d\xi}{ds}\right)^2 + \xi^2\right) ds. \tag{15}$$

After integration and the substitution

$$b = \left(\xi - \xi\left(0\right)\cosh\left(s\right)\right)/\sinh\left(s\right), \tag{16}$$

one finds

$$S_0\left(\xi,s\right) = -\left(1/2\right)\left(\left(\xi^2 + \xi\left(0\right)^2\right)\cosh\left(s\right) - 2\xi\xi\left(0\right)\right)/\sinh\left(s\right). \tag{17}$$

The covariant Schrödinger type equation for a spin zero particle is

$$\hat{\mathcal{H}}|\Psi\left(s\right)\rangle = \frac{\left(\hat{p} + eA\right)^2}{2}|\Psi\left(s\right)\rangle = i\frac{\partial|\Psi\left(s\right)\rangle}{\partial s}, \tag{18}$$

and the proper time evolved state is found from

$$|\Psi(s)\rangle = \hat{U}(s)|\Psi(0)\rangle = e^{-i\hat{\mathcal{H}}s}|\Psi(0)\rangle, \tag{19}$$

The covariant proper time Green's function is

$$\mathcal{G}(\vec{q}, q', s) = \langle q|\hat{U}(s)|q'\rangle\theta(s), \tag{20}$$

where $\theta(s)$ is the Heaviside function. This function is a solution to the partial differential equation

$$\left(\hat{\mathcal{H}} - i\frac{\partial}{\partial s}\right)\mathcal{G}(q, q', s) = -i\delta(q - q')\delta(s), \tag{21}$$

and it satisfies the condition

$$\lim_{s\to 0}\mathcal{G}(q, q', s) = \delta(q - q'). \tag{22}$$

The covariant Hamiltonian operator for the accelerated particle is

$$\hat{\mathcal{H}} = \left(\left(\hat{p}^0 + z\right)^2 - \hat{\vec{p}}\cdot\hat{\vec{p}}\right)/2, \tag{23}$$

with $\hat{p}_\mu = i\partial/\partial q^\mu$, which satisfy the commutation relation

$$[\hat{q}^\mu, \hat{p}^\nu] = -ig^{\mu\nu}. \tag{24}$$

The s-dependent operators are

$$\begin{aligned} \hat{\xi}(s) &= \hat{U}(s)\hat{\xi}\hat{U}(-s) = \hat{\xi}\cosh(s) + \hat{p}^3\sinh(s) \\ \hat{t}(s) &= \hat{U}(s)\hat{t}\hat{U}(-s) = t + \hat{\xi}\sinh(s) + \hat{p}^3(\cosh(s) - 1), \end{aligned} \tag{25}$$

and the solutions to the classical equations of motion Eq. (13) are found from the matrix elements $\langle\Psi(0)|\hat{\xi}(s)|\Psi(0)\rangle$, and $\langle\Psi(0)|\hat{t}(s)|\Psi(0)\rangle$, where $|\Psi(0)\rangle$ is a minimum uncertainty state [1]. Using the identities

$$\begin{aligned} \langle q|\hat{U}(s)\hat{\xi}\hat{U}(-s)\hat{U}(s)|q'\rangle &= \left(\hat{p}^0 + \hat{z}'\right)\langle q|\hat{U}(s)|q'\rangle \\ \langle q|\hat{U}(s)\hat{t}\hat{U}(-s)\hat{U}(s)|q'\rangle &= \hat{t}'\langle q|\hat{U}(s)|q'\rangle, \end{aligned} \tag{26}$$

one can show that the Green's function also satisfies the first order partial differential equations

$$\begin{aligned} \left(\left(\hat{p}^0 + \hat{z}\right)\cosh(s) + \hat{p}^3\sinh(s)\right)\mathcal{G}(q, q', s) &= \left(\hat{p}^0 + z'\right)\mathcal{G}(q, q', s) \\ \left(\hat{t} + \left(\hat{p}^0 + \hat{z}\right)\sinh(s) + \hat{p}^3(\cosh(s) - 1)\right)\mathcal{G}(q, q', s) &= t'\mathcal{G}(q, q', s). \end{aligned} \tag{27}$$

The solution to the equations Eq. (21) and Eq. (27) can be found using a method inspired by Dirac [5] for non-relativistic quantum mechanics. One assumes a solution of the form

$$\langle q|\hat{U}(s)|q'\rangle = a_0(s)\int_{-\infty}^{+\infty} e^{i\Phi}dp^0, \tag{28}$$

with

$$\Phi = S_0(z, z', p0, s) - \frac{(y - y')^2}{2s} - \frac{(x - x')^2}{2s} - p^0(t - t'), \tag{29}$$

and $q'^\mu = q^\mu(0)$. The evaluation of Eq. (28) and substitution into Eq. (21) leads to the differential equation for $a_0(s)$,

$$\frac{da_0(s)}{ds} = -\left(1/s + \coth(s/2)/2\right)a_0(s). \tag{30}$$

Solving this equation and using the condition Eq. (22) gives the final result

$$\mathcal{G}\left(q,q',s\right)=\frac{-i\theta\left(s\right)e^{i\left(-B^{2}A+C\right)}e^{-i\frac{\left(x-x'\right)^{2}}{2s}}e^{-i\frac{\left(y-y'\right)^{2}}{2s}}}{8\pi^{2}s\sinh\left(s/2\right)}, \tag{31}$$

with

$$\begin{aligned}A\left(s\right)&=-\tanh\left(s/2\right)\\ B\left(t,t',z,z',s\right)&=\frac{1}{2}\left(z+z'+\left(t-t'\right)\coth\left(s/2\right)\right)\\ C\left(z,z',s\right)&=\left(-\cosh\left(s\right)\frac{1}{2}\left(z^{2}+z'^{2}\right)+zz'\right)\sinh^{-1}\left(s\right).\end{aligned} \tag{32}$$

The evolved wave function $\Psi(q,s)$ is found from the initial state wave function $\Psi(q,0)$ using the evolution equation

$$\Psi\left(q,s\right)=\int\mathcal{G}\left(q,q',s\right)\Psi\left(q',0\right)dq'. \tag{33}$$

Examples of evolved wave functions for a free particle are found in [1]. In these examples the sum of the exponents of the initial state wave function and the Green's function are combined by completing the square, and the q' integration is then performed. For a Gaussian initial state, it is found that the evolved free particle wave function spreads about the classical path described by the coordinates $q^{\mu}(s)=(p^{0}s,\vec{p}s)/m$, with $s=t/\gamma$. For plane wave, and Gaussian initial state wave functions, the integration using Eq. (31) can be performed in a similar manner. Normal units may be restored in Eq. (31) with the replacements $q^{\mu}\rightarrow q^{\mu}/q_{s}$, $p^{\mu}\rightarrow p^{\mu}/p_{s}$, and $s\rightarrow s/s_{s}$, with $q_{s}=\sqrt{\hbar c/(mg)}$, $q_{s}p_{s}=\hbar$, and $s_{s}=c^{2}/g$. In the limit $g\rightarrow 0$, the Green's function Eq. (31) becomes the free particle Green's function,

$$\mathcal{G}\left(q,q',s\right)\rightarrow\frac{-i\left(mc\right)^{2}}{4\left(\pi\hbar s\right)^{2}}e^{i\frac{mc\left(q-q'\right)^{2}}{2s\hbar}}. \tag{34}$$

Calculations have been confirmed with the aid of REDUCE [6].

---

* Email address: bronco@stp.dias.ie
† Also *Institiúid Teicneolaíochta Bhaile Átha Cliath.*

1. T. Garavaglia, *Covariant relativistic quantum theory*, hep-th/9907059.
2. H. Goldstein, *Classical Mechanics* (Addison Wesley, Reading Mass., 1959) p. 207.
3. V. Fock, Physik Z. Sowjetunion **12**, 404 (1937).
4. J. Schwinger, Phys. Rev. **82**, 664 (1951).
5. P. A. M. Dirac, Physikalische Zeitschrift der Sowjetunion, Band **3**, Heft **1**, 64 (1933).
6. A. C. Hearn, and J. P. Fitch, *REDUCE User's Manual*, (Rand, Santa Monica, 1995).